ON THE SPECTRUM OF PERTURBATIONS OF PLANE PARALLEL FLOWS AT LOW REYNOLDS NUMBERS

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An extensive literature is devoted to the investigation of the behavior of perturbations of plane parallel steady flows (see [1 and 2]). Basic attention is usually given to the determination of the stability of these flows and calculation of the critical values of the Reynolds number. Meanwhile, for the construction of a nonlinear stability theory, for the investigation of the behavior of distrurbances of arbitary form, and also for the solution of a number of other problems, a knowledge is required not only of the critical perturbations, but also the whole spectrum of normal perturbations in the entire range of variation of the two parameters Reynolds number and wave number.

For large values of Reynolds number the spectrum of perturbations can be investigated by an asymptotic method. In the paper of Grohne [3] results are given of such an investigation for Couette and Poiseuille flow and discussion is given of the dependence of certain "lower" decrements on the Reynolds number for fixed values of the wave number.

In the present paper we construct and use expansions of the normal perturbations and their decrements in power series in the Reynolds number R These expansions permit us to establish the characteristic features of the spectrum of perturbations for small Reynolds number for an arbitrary velocity profile.

In a quiescent fluid ($R = 0$) the perturbations decay monotonously (all decrements are real and positive). For $R \neq 0$ the form of the spectrum is qualitatively different in the cases of flows with even and odd profiles. In flows with even profiles the decrements are complex for arbitrarily small R, that is the perturbations are only "running" perturbations, whose phase velocity increases with R. In the case of a flow with an odd profile the expansions of the decrements are found to be real ("standing" perturbations); however in this case there is on the R-axis a singular point R_* and the expansions are correct only up to it. At that point the decrements merge with the corresponding perturbation of the other parity, and for $R > R_*$ complex conjugate decrements occur. Simple intersections in the spectrum are impossible, as is shown.

As examples the decrements for some lower levels of the spectrum are calculated in the paper by the method of perturbations at arbitrary values of the wave number for Poiseuille and Couette flows and a flow with a cubic velocity profile.

1. Normal perturbations. Orthogonality conditions. We consider plane parallel steady motion of a fluid between the planes $x = \pm h$. We choose as unit of velocity a characteristic velocity U_0 of the steady motion, and as units of length and time we choose h and h^2/ν, respectively (where ν is the kinematic viscosity). The stream function of small perturbations of the steady motion satisfies a linear equation with coefficients not depending upon z and t (where z is the coordinate along the flow and t the time). Consequently, there exist normal plane perturbations of the form $\varphi(x) \exp(-\lambda t + i\alpha z)$, where α is a real wave number, and $\varphi(x)$ and λ are the amplitude and decrement of the perturbation. The decrement λ is related to the complex phase velocity of the perturbation $c = c_r + ic_i$ by the relation $\lambda = i\alpha c$, that is

$$\operatorname{Re}\lambda = -\alpha c_i, \qquad \operatorname{Im}\lambda = \alpha c_r \tag{1.1}$$

The amplitude $\varphi(x)$ satisfies the Orr-Sommerfield equation

$$\left(U - \frac{c}{R}\right)(\varphi'' - \alpha^2\varphi) - U''\varphi = \frac{1}{i\alpha R}(\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi) \qquad \left(R = \frac{U_0 h}{\nu}\right) \tag{1.2}$$

and the conditions on solid boundaries

$$\varphi = \varphi' = 0 \quad \text{for } x = \pm 1 \tag{1.3}$$

Here $U(x)$ is the velocity of the steady motion, and R is the Reynolds number. We introduce in place of the phase velocity c the decrement λ, and rewrite (1.2) in the form

$$L\varphi \equiv rH\varphi - \Delta^2\varphi = \lambda\Delta\varphi \qquad \left(r = i\alpha R,\ \Delta\frac{d^2}{dx^2} - \alpha^2\right) \tag{1.4}$$

Here H is an operator depending upon the profile $U(x)$

$$H\varphi = U\Delta\varphi - U''\varphi \tag{1.5}$$

Equation (1.4) and the boundary conditions (1.3) determine an infinite sequence of normal perturbations $\varphi_i(x)$ and decrements λ_i. The operator L in (1.4) is not self-adjoint, and its eigenvalues λ_i and eigenfunctions φ_i are in general complex.

We consider an adjoint boundary-value problem. To find the form of the adjoint operator we multiply the equation that is the complex conjugate of (1.4) by the function $\psi(x)$, which satisfies the boundary conditions

$$\psi = \psi' = 0 \quad \text{for } x = \pm 1 \tag{1.6}$$

and integrate with respect to x from -1 to $+1$. Interchanging derivatives of φ^* and ψ through integration by parts, and equating to zero the factor of φ^*, we obtain

$$L^+\psi \equiv -rH^+\psi - \Delta^2\psi = \lambda^*\Delta\psi, \qquad H^+\psi = \Delta(U\psi) - U''\psi \tag{1.7}$$

Equation (1.7) and conditions (1.6) determine an infinite sequence of conjugate normal perturbations $\psi_i(x)$ and decreaments λ_i^*.

The perturbations φ_i and conjugate perturbations ψ_k belonging to different decrements λ_i and λ_k^* are orthogonal in a definite sence. From

(1.4) and (1.7) can be obtained the relation

$$(\lambda_i - \lambda_k) \int \psi_k^* \Delta\varphi_i dx = 0 \tag{1.8}$$

(here and henceforth integration with respect to x is carried between the limits from -1 to $+1$). From (1.8) with $\lambda_i \neq \lambda_k$ follows the orthogonality condition

$$\int \psi_k^* \Delta\varphi_i dx = 0 \tag{1.9}$$

2. Perturbations in a quiescent fluid. We consider first the spectrum of perturbations in a quiescent fluid ($R = 0$). In this case the operator L is self-adjoint. For real normal perturbations $\varphi_i{}^{0}$ and the conjugate perturbations $\psi_i{}^{(0)}$ corresponding to them we will have Equation

$$\Delta^2\varphi^{(0)} = -\lambda^{(0)}\Delta\varphi^{(0)} \tag{2.1}$$

and boundary conditions

$$\varphi^{(0)} = \varphi^{(0)\prime} = 0 \qquad \text{for } x = \pm 1 \tag{2.2}$$

From (2.1) follows the orthogonality condition

$$\int \varphi_i{}^{(0)} \Delta\varphi_k{}^{(0)} dx = 0 \qquad (i \neq k) \tag{2.3}$$

For $i = k$ the integral in (2.3) is always negative, and consequently the eigenfunctions $\varphi_i{}^{(0)}$ can be normalized to -1. The orthogonality condition takes the form

$$\int \varphi_i{}^{(0)} \Delta\varphi_k{}^{(0)} dx = -\delta_{ik} \tag{2.4}$$

From (2.1) it is easy to obtain the relation

$$\lambda^{(0)} = -\int \varphi^{(0)} \Delta^2\varphi^{(0)} dx \Big/ \int \varphi^{(0)} \Delta\varphi^{(0)} dx \tag{2.5}$$

The integral in the numerator of (2.5) is always positive, but the normalizing integral in the denominator is negative. Thus the eigenvalues $\lambda^{(0)}$ are real and positive — all perturbations in a quiescent fluid decay monotonously.

The problem (2.1) - (2.2) has even and odd solutions. The normalized even eigenfunctions are

$$\varphi_i{}^{(0)} = \frac{1}{\sqrt{J_i}} \left[\frac{\cosh \alpha x}{\cosh \alpha} - \frac{\cos\sqrt{\lambda_i{}^{(0)} - \alpha^2}\, x}{\cos\sqrt{\lambda_i{}^{(0)} - \alpha^2}}\right] \qquad (i = 0, 2, 4, \ldots) \tag{2.6}$$

where

$$J_i = \frac{\lambda_i{}^{(0)}}{2(\alpha^2 - \lambda_i{}^{(0)})} (\alpha^2 + \alpha \tanh \alpha - \alpha^2 \tanh^2 \alpha - \lambda_i{}^{(0)})$$

The decrements of the even parturbations are determined by Equation

$$\sqrt{\lambda_i{}^{(0)} - \alpha^2} \tan \sqrt{\lambda_i{}^{(0)} - \alpha^2} = -\alpha \tanh \alpha \tag{2.7}$$

and depend upon the wave number α. For $\alpha = 0$

$$\lambda_i{}^{(0)} = {}^1/_4 (i + 2)^2 \pi^2 \qquad (i = 0, 2, 4, \ldots) \tag{2.8}$$

With increasing α the decrements $\lambda_i{}^{(0)}(\alpha)$ of the even perturbations at first decrease, pass through a minimum, and then increase, so that for

$\alpha \gg 1$

$$\lambda_i^{(0)} = {}^1/_4\,(i+1)^2\,\pi^2 + \alpha^2 \tag{2.9}$$

The odd eigenfunctions are

$$\varphi_i^{(0)} = \frac{1}{\sqrt{J_i}}\left[\frac{\sinh \alpha x}{\sinh \alpha} - \frac{\sin\sqrt{\lambda_i^{(0)}-\alpha^2}\,x}{\sin\sqrt{\lambda_i^{(0)}-\alpha^2}}\right] \qquad (i = 1, 3, 5, \ldots) \tag{2.10}$$

Here

$$J_i = \frac{\lambda_i^{(0)}}{2\,(\alpha^2-\lambda_i^{(0)})}\,(\alpha^2 + \alpha\coth\alpha - \alpha^2\coth^2\alpha - \lambda_i^{(0)})$$

The decrements of the odd perturbations are found from Equation

$$\sqrt{\lambda_i^{(0)}-\alpha^2}\,\cot\sqrt{\lambda_i^{(0)}-\alpha^2} = \alpha\coth\alpha \tag{2.11}$$

For $\alpha = 0$ the decrements are the roots of Equation $\sqrt{\lambda_i^{(0)}}\cot\sqrt{\lambda_i^{(0)}} = 1$ and are: $\lambda_1^{(0)} = 20.191$, $\lambda_3^{(0)} = 59.680$, $\lambda_5^{(0)} = 118.90$, ... (see [4]). With increasing α all odd decrements increase monotonously, and for $\alpha \gg 1$ Equation (2.9) is valid with $i = 1, 3, 5, \ldots$ For all values of the wave number α the even and odd decrements $\lambda_i^{(0)}$ alternate with increase of the index i; their sequence in increasing order is $\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots$

The complete orthonormal system of base functions (2.6) and (2.10) is conveniently used for approximate solution of the Orr-Sommerfeld equation in various variants of the perturbation method, and also for reduction of that equation to a system of algebraic equations with the use of an electronic computer. In [5] the even subsystem of functions (2.6) was used for numerical calculation of the stability of plane Poiseuille flow.

3. Expansion of powers of Reynolds number. In the previous section we considered perturbations and their decrements in a quiescent fluid ($R = 0$). For small values of Reynolds number we can seek the solution of the problem in the form of a series in powers of the parameters $r = i\alpha R$. We set

$$\varphi = \varphi^{(0)} + r\varphi^{(1)} + r^2\varphi^{(2)} + \ldots, \qquad \lambda = \lambda^{(0)} + r\lambda^{(1)} + r^2\lambda^{(2)} + \ldots \tag{3.1}$$

Substituting the series (3.1) into Equation (1.4) and equating terms with like powers of r we obtain the equations of a sequence of approximations

$$\begin{aligned}
&\Delta^2\varphi^{(0)} + \lambda^{(0)}\Delta\varphi^{(0)} = 0\\
&\Delta^2\varphi^{(1)} + \lambda^{(0)}\Delta\varphi^{(1)} = -\lambda^{(1)}\Delta\varphi^{(0)} + H\varphi^{(0)}\\
&\Delta^2\varphi^{(2)} + \lambda^{(0)}\Delta\varphi^{(2)} = -\lambda^{(2)}\Delta\varphi^{(0)} - \lambda^{(1)}\Delta\varphi^{(1)} + H\varphi^{(1)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\Delta^2\varphi^{(n)} + \lambda^{(0)}\Delta\varphi^{(n)} = -\lambda^{(n)}\Delta\varphi^{(0)} - \lambda^{(n-1)}\Delta\varphi^{(1)} + \ldots - \lambda^{(1)}\Delta\varphi^{(n-1)} + H\varphi^{(n-1)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{3.2}$$

$$\varphi^{(n)} = \varphi^{(n)\prime} = 0 \quad \text{for } x = \pm 1 \tag{3.3}$$

The first of equations (3.2) (the zeroth approximation), representing perturbations in a quiescent fluid, was considered above. Corrections to $\varphi^{(0)}$ are found from equations of the form

$$\Delta^2\varphi^{(n)} + \lambda^{(0)}\Delta\varphi^{(n)} = f_n(x) \tag{3.4}$$

Here the $f_n(x)$ are known functions, depending on the preceeding approximations and containing the $\lambda^{(n)}$. Multiplying (3.4) by $\varphi^{(0)}$ and integrating with respect to x, we obtain the conditions of solvability of (3.4)

$$\int f_n(x)\,\varphi^{(0)}\,dx = 0 \tag{3.5}$$

from which are found the corrections to the decrements $\lambda^{(n)}$. Then solving the nonhomogeneous equations (3.4) with the boundary conditions (3.3) we find the $\varphi^{(n)}$. In this way we can determine in succession the coefficients of the expansions (3.1). It is easy to see that all the $\varphi^{(n)}$ and $\lambda^{(n)}$ are real, and the series (3.1) may be written, separating real and imaginary parts, in the form

$$\varphi = (\varphi^{(0)} - \alpha^2R^2\varphi^{(2)} + \alpha^4R^4\varphi^{(4)} - \ldots) + i\alpha R\,(\varphi^{(1)} - \alpha^2R^2\varphi^{(3)} + \\ + \alpha^4R^4\varphi^{(5)} - \ldots) \tag{3.6}$$

$$\lambda = (\lambda^{(0)} - \alpha^2R^2\lambda^{(2)} + \alpha^4R^4\lambda^{(4)} - \ldots) + i\alpha R\,(\lambda^{(1)} - \alpha^2R^2\lambda^{(3)} + \\ + \alpha^4R^4\lambda^{(5)} - \ldots) \tag{3.7}$$

From (3.5) with consideration of the normalization condition (2.4) we find

$$\lambda^{(1)} = -\int \varphi^{(0)} H\varphi^{(0)}dx, \quad \lambda^{(2)} = \lambda^{(1)}\int \varphi^{(0)}\Delta\varphi^{(1)}dx - \int \varphi^{(0)}H\varphi^{(1)}dx \\ \lambda^{(3)} = \lambda^{(2)}\int \varphi^{(0)}\Delta\varphi^{(1)}dx + \lambda^{(1)}\int\varphi^{(0)}\Delta\varphi^{(2)}dx - \int\varphi^{(0)}H\varphi^{(2)}dx \tag{3.8}$$

. . . .

An expansion in powers of the parameter r can be constructed also for the solution of the adjoint problem (1.7). In so doing one uses equations analogous to those given above, with r replaced by $-r$ and H by H^+.

The sequence of approximations is conveniently found by use of the method of perturbations, expanding $\varphi^{(n)}$ in the basic system of eigenfunctions of the unperturbed problem (2.1)

$$\varphi_i^{(n)} = \sum_k c_{ik}^{(n)}\,\varphi_k^{(0)} \tag{3.9}$$

We give the equations for the corrections of first and second order to the eigenvalues and eigenfunctions of the ith level

$$\lambda_i^{(1)} = -H_{ii}, \qquad \varphi_i^{(1)} = c_{ii}^{(1)}\varphi_i^{(0)} + \sum_{k\neq i}\frac{H_{ki}}{\lambda_k^{(0)} - \lambda_i^{(0)}}\,\varphi_k^{(0)} \tag{3.10}$$

$$\lambda_i^{(2)} = \sum_{k\neq i}\frac{H_{ik}H_{ki}}{\lambda_i^{(0)} - \lambda_k^{(0)}}$$

$$\varphi_i^{(2)} = c_{ii}^{(2)}\varphi_i^{(0)} - \sum_{k\neq i}\frac{H_{ki}}{\lambda_i^{(0)} - \lambda_k^{(0)}}\left[c_{ii}^{(1)} + \frac{H_{ii}}{\lambda_i^{(0)} - \lambda_k^{(0)}}\right]\varphi_k^{(0)} + \\ + \sum_{k\neq i}\sum_{l\neq i}\frac{H_{kl}H_{li}}{(\lambda_i^{(0)} - \lambda_k^{(0)})(\lambda_i^0 - \lambda_l^{(0)})}\,\varphi_k^{(0)} \tag{3.11}$$

Here

$$H_{ik} = \int \varphi_i^{(0)} H \varphi_k^{(0)}\, dx \tag{3.12}$$

The summations in Equations (3.10) and (3.11) run over the unperturbed levels. The coefficients $c_{ii}^{(1)}$ and $c_{ii}^{(2)}$ are determined by normalization.

The expansions obtained in this section are valid up to a singular point. As will be evident from what follows, the existence of this singular point is connected with the symmetry of the profile $U(x)$ of the steady motion. The expansions are found to be essentially different in the cases of flows with even and odd velocity profiles. Henceforth these cases will be considered separately.

4. Flows with even profiles. If the profile of the steady flow has the symmetry property $U(-x) = U(x)$ (an example would be plane Poiseuille flow), then the arbitrary values of the parameter r there exist even and odd solutions of the problem (1.3), (1.4). Thus the perturbation spectrum decomposes into two independent systems o even and odd levels. The parity of any of the levels at any value of Reynolds number is determined by the parity at $r = 0$. From the equations for successive approximations (3.2) it follows that for an even $U(x)$, and consequently an even operator H, all corrections $\varphi^{(n)}$ have just the same parity as the zeroth approximation $\varphi^{(0)}$. As is evident from Equations (3.8), all the $\lambda^{(n)}$, are in this case generally different from zero, and consequently the decrement λ is complex for arbitrarily small r. Thus in the flow with an even profile the velocity perturbations are only "running" perturbations, whose phase velocity

$$c_r = \alpha^{-1} \operatorname{Im} \lambda = R\,(\lambda^{(1)} - \alpha^2 R^2 \lambda^{(3)} + \ldots) \tag{4.1}$$

is, for small values of the Reynolds number, proportional to R.

As an example we consider plane Poiseuille flow with the parabolic profile $U = 1 - x^2$. We find the corrections $\lambda^{(1)}$ and $\lambda^{(2)}$ of first and second order to the decrement by using the theory perturbations.

The matrix elements of the perturbation operator H_{ik} appearing in (3.10), (3.11) are different from zero only for equal parity of the indices. If i and k are even, then

$$H_{ik} = \frac{1}{\sqrt{J_i J_k}} \left\{ \left[\frac{1}{\alpha^2} + \frac{4\lambda_i^{(0)} (\lambda_i^{(0)} - 2\lambda_k^{(0)})}{\lambda_k^{(0)} (\lambda_k^{(0)} - \lambda_i^{(0)})^2} \right] (\alpha^2 + \alpha \tanh \alpha - \alpha^2 \tanh^2 \alpha) + \right. \tag{4.2}$$

$$\left. + \frac{2\lambda_i^{(0)} (3\lambda_k^{(0)} - \lambda_i^{(0)})}{(\lambda_i^{(0)} - \lambda_k^{(0)})^2} \right\} \qquad (i \neq k)$$

$$H_{ii} = \frac{1}{J_i} \left\{ \left(\frac{1}{\alpha^2} + \frac{1}{a_i^2} + \frac{4}{\lambda_i^{(0)}} + \frac{\lambda_i^{(0)}}{4a_i^4} \right) (\alpha^2 + \alpha \tanh \alpha - \alpha^2 \tanh^2 \alpha) - \right. \tag{4.3}$$

$$\left. - \left(1 + \frac{\lambda_i^{(0)}}{3} + \frac{\alpha^2}{a_i^2} + \frac{\lambda_i^{(0)}}{4a_i^2} + \frac{\alpha^2 \lambda_i^{(0)}}{4a_i^4} \right) + \frac{\lambda_i^{(0)}}{3a_i^2} \tanh^2 \alpha \right\} \qquad (a_i^2 = \alpha^2 - \lambda_i^{(0)})$$

In the case of odd i and k the equations for the matrix elements are obtained from (4.2) and (4.3) by replacing $\tanh\alpha$ by $\coth\alpha$.

Substituting the H_{ik} into equations of perturbation theory (3.10) and (3.11) we obtain the $\lambda_i^{(1)}$ and $\lambda_i^{(2)}$ as functions of the wave number α. The series of perturbation theory, which it is necessary to sum by calculation of the quadratic corrections to the decrement $\lambda_i^{(2)}$, converge sufficiently rapidly. The rapidity of convergence is evident, for example, from the formulas for $\lambda_i^{(2)}$ with $\alpha = 0$ and $\alpha \gg 1$

$$\lambda_i^{(2)} = \frac{2^{10}}{\pi^6} \sum_{k \neq i} \frac{[(2+i)^2 + (2+k)^2]^2}{[(2+i)^2 - (2+k)^2]^5} \qquad (\alpha = 0)$$

$$\lambda_i^{(2)} = \frac{2^{12}}{\pi^6} \sum_{k \neq i} \frac{(i+1)^2 (k+1)^2}{(i-k)^5 (i+k+2)^5} \qquad (\alpha \gg 1)$$

(where i is even; summation carried over even values of k).

Fig.1 shows $\lambda_0^{(1)}$ and $\lambda_2^{(1)}$ as functions of α. The corrections $\lambda_1^{(1)}$ and $\lambda_3^{(1)}$ to the odd levels are near to 0.7 and practically do not depend upon α ($\lambda_3^{(1)} > \lambda_1^{(1)}$).(*) Fig.2 shows the quadratic corrections $\lambda_i^{(2)}$ to the decrement for the "lowest" four levels. From the graphs it is seen that some of the quadratic corrections change sign with varying α. As is clear from the expansion (3.7), the sign of $\lambda^{(2)}$, indicates the increase or decrease of stability with increasing R, $\lambda^{(2)} > 0$ corresponding to a reduction and $\lambda^{(2)} < 0$ to an increase of stability compared with $R = 0$.

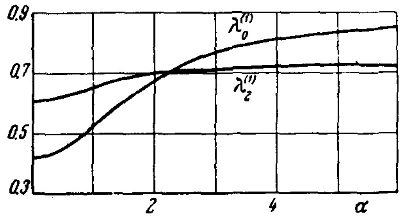

Fig. 1

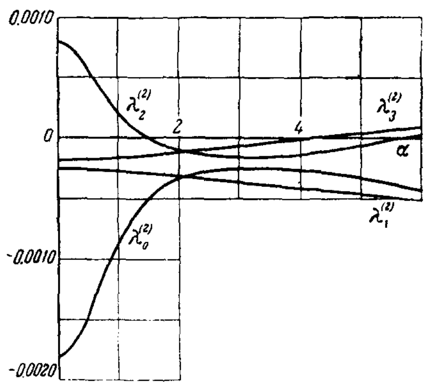

Fig. 2

5. Flows with odd profiles. We consider now a steady flow with an odd profile: $U(-x) = -U(x)$. As a consequence of the oddness of the operator H the solutions of the problem (1.3), (1.4) for any arbitrary $r \neq 0$ do not possess a definite parity.

We consider first of all the structure of the matrix of the operator L in (1.4) in the case of odd H. Choosing as a basis the functions $\varphi_n^{(0)}$ of (2.6) to (2.9), we rewrite (1.4) in matrix form

$$\sum_n c_n [(\lambda - \lambda_n^{(0)}) \delta_{mn} + i\alpha R H_{mn}] = 0 \qquad (5.1)$$

where the c_n are the coefficients of the expansion of the perturbation φ on the basis of the functions $\varphi_n^{(0)}$, and the matrix elements H_{mn} of the operator H are different from zero only for indices of different parity. By

*) The first-order corrections $\lambda_0^{(1)}$ and $\lambda_1^{(1)}$ were calculated previously in [6].

a unitary transformation the matrix corresponding to Equation (5.1) can be reduced to the real form

$$(\lambda - \lambda_n^{(0)}\, \delta^{mn} + (-1)^n \alpha R H_{mn}) \tag{5.2}$$

and consequently the eigenvalues of Equation (5.1) are either real or form complex conjugate pairs.

We turn to the equations of the successive approximations (3.2) and (3.8). From the first equation of the system (3.8) it follows that $\lambda^{(1)} = 0$. Then the second equation of the system (3.2) leads to the conclusion that the correction $\varphi^{(1)}$ has parity opposite to $\varphi^{(0)}$. The condition of solvability of the equation for $\varphi^{(2)}$ permits determination of $\lambda^{(2)}$, which is, generally speaking, different from zero; then the functions $\varphi^{(2)}$ have the same parity as $\varphi^{(0)}$. Progressing further in the system (3.2) and the conditions of solvability (3.8), it is easily seen that all odd corrections to the decrement vanish: $\lambda^{(1)} = \lambda^{(3)} = \ldots = 0$, and consequently the functions $\varphi^{(n)}$ have alternating parity.

Thus in flows with odd velocity profiles the expansion of the decrement λ in powers of Reynolds number is found to be real

$$\lambda = \lambda^{(0)} - \alpha^2 R^2 \lambda^{(2)} + \alpha^4 R^4 \lambda^{(4)} - \ldots \tag{5.3}$$

Hence it follows that for small Reynolds number the perturbations of odd flows are monotonous; their phase velocity $c_r = 0$ ("stabding" perturbations). Furthermore, it is evident from the expansion (3.6) that the real part of the perturbation φ has the same parity as $\varphi^{(0)}$ (the perturbation for $R = 0$), and the parity of the imaginary part of φ is opposite to that of $\varphi^{(0)}$. Thus the expansions (3.6) arising from even and odd levels $\varphi_i^{(0)}$ at $R = 0$ have different forms. If $\varphi_i^{(0)}$ is real eigenfunction of an even level ($i = 0, 2, 4, \ldots$) at $R = 0$, then with increasing R there appears an imaginary odd part proportional to R for small R. For odd $\varphi_i^{(0)}$ ($i = 1, 3, 5, \ldots$) there appears for $R \neq 0$ an even imaginary part. It is possible by convention to call the expansions of φ_i "even" for $i = 0, 2, 4, \ldots$ and "odd" for $i = 1, 3, 5, \ldots$, and for $R \neq 0$, since for $R \to 0$, they reduce respectively to the even and odd functions $\varphi_i^{(0)}$

The conclusions obtained regarding the reality of the decrements and the form of the expansions for the eigenfunctions are valid naturally only up to the singular point on the R-axis. That such a singular point actually exists is indicated, for example, by the results of investigations of the perturbation spectrum for plane Couette flow (cf. [3 and 7]), from which arises the existence, beginning at a certain Reynolds number, of perturbations with complex decrements. It will be shown below that the appearance of oscillating perturbations is connected with the intersection of the "even" and "odd" levels.

We obtain a necessary condition for the appearance of oscillating perturbations which arises directly from the property of oddness of the profile $U(x)$. For this purpose we shall devide the solutions of the basic and adjoint problems into even (subscript g) and odd (subscript u) parts

$$\varphi = \varphi_g + \varphi_u, \qquad \psi = \psi_g + \psi_u \tag{5.4}$$

Multiplying Equation (1.4) in turn by ψ_g and ψ_u, and (1.7) by φ_g and φ_u and integrating with respect to x, we obtain four integral relations. From these relations, with consideration of the oddness of the operators H and H^+ and the connection between them, follows

$$(\lambda^* - \lambda)\int[\psi_u \Delta\varphi_u - \psi_g \Delta\varphi_g]\, dx = 0 \tag{5.5}$$

The integral in (5.5) will henceforth be denoted by I. From (5.5) follows a necessary condition for the appearance of oscillating perturbations ($\lambda^* \neq \lambda$), the vanishing of the integral I. Untill then the expansions (3.6) and (3.7) are valid, the decrements are real ($\lambda^* = \lambda$), and the real integral I is different from zero and has different signs for the "even" and "odd" levels. Indeed for $R \to 0$ the functions (5.4) reduce to functions of definite parity, and by virtue of the normalizing conditions (2.4) we have $I = \pm 1$, where the plus and minus correspond respectively to the even and odd levels. (We recall that the conjugate solutions for $R = 0$ were chosen to coincide with the basic ones).

Thus for any of the normal perturbations ("even" or "odd") the integral $I \neq 0$ for $R = 0$ and, consequently, also for small R. Hence the vanishing of I and the appearance of oscillating perturbations connected with it occur at some finite value R_* of the Reynolds number, and also simultaneously there should appear a pair of normal perturbations corresponding to the complex conjugate decrements λ and λ^*; that is, at the point R_* should occur the confluence of two real decrements.

The behavior of the decrements near the point of confluence can be traced by means of an approximate method used in quantum mechnics for the investigation of molecular terms ([8], Section 79) (*).

Let R_0 be the value of the Reynolds number for which two neighboring decrements λ_1 and λ_2 are real and close (but not coincident). We denote by φ_1 and φ_2 the solutions corresponding to λ_1 and λ_2, and by ψ_1 and ψ_2 the conjugate solutions at point R_0. The solution at a nearby point $R_0 + \delta R$ is found from Equation

$$i\alpha\,(R_0 + \delta R)\, H\varphi - \Delta^2\varphi = \lambda\Delta\varphi \tag{5.6}$$

and can be approximately represented in the form

$$\varphi = c_1\varphi_1 + c_2\varphi_2 \tag{5.7}$$

Substituting (5.7) into (5.6), multiplying in turn by ψ_1^* and ψ_2^*, and integrating, we obtain a system of linear homogeneous equations for the coefficients c_1 and c_2 of the expansion. From the compatibility condition we

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*) This method was used for the investigation of the intersection of levels in the perturbation spectrum of equilibrium of a conducting fluid in a mag-netic field [9].

can determine the decrements near R_0

$$\lambda = {}^1/_2\,[\lambda_1 + \lambda_2 + (V_{11} + V_{22})\,\delta R\,] \pm \pm \sqrt{{}^1/_4\,[\lambda_1 - \lambda_2 + (V_{11} - V_{22})\,\delta R\,]^2 + V_{12}V_{21}\,(\delta R)^2} \qquad (5.8)$$

Here

$$V_{mn} = i\alpha I_n^{-1}\int \psi_m{}^* H\varphi_n dx, \qquad I_n = \int \psi_n{}^* \Delta\varphi_n dx \qquad (m, n = 1, 2) \quad (5.9)$$

where the matrix elements V_{11} and V_{22} are real, and V_{12} and V_{21} are imaginary (cf. the expansion (3.6) and Equations (5.4)). The coordinate of the point of intersection is $R_* = R_0 + \delta R$, where δR is determined from condition $\lambda_+ = \lambda_-$.

From Equation (5.8) it is evident first of all that a simple intersection is not possible, in which both decrements are real on both sides of the point of intersection. For such an intersection it is evidently necessary that $V_{12}V_{21} = 0$ identically in R_0, which obviously does not occur. In the case $V_{12}V_{21} > 0$ intersection is impossible. The necessary condition for an intersection is $V_{12}V_{21} < 0$; then at point R_* there occurs a conjunction of real decrements, wheras for $R > R_*$ the two decrements are complex conjugates. As is easily seen, the solutions φ_+ and φ_- of (5.7) coincide at point R_* , and at that point both the normalized integrals I_+ and I_- in (5.5) vanish. Thus the system of normal perturbations ceases to be complete at the point R_* ; for completeness it is necessary to add a "supplementary" solution [10] .

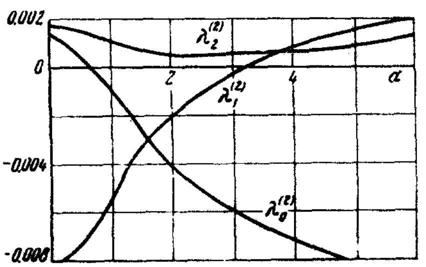

Fig. 3

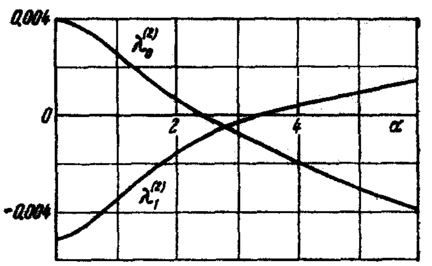

Fig. 4

Thus neighboring real decrements (which consequently have different parities) either never intersect at all, or merge in certain points R_* with the formation of complex conjugate pair. Since in the real domain the spectrum has alternating "parity", this conclusion is valid with respect to any two levels of different "parity"; but the intersection of the same "parity" is not at all possible.

6. Examples of flows with odd profile. As examples we consider Couette flow with the linear profile $U = x$ and the flow with a cubic profile $U = x(1 - x^2)$. (Flow of this form arises in steady convection of a fluid between vertical parallel plates heated to different temperatures [11]).

For Couette flow the matrix elements are

$$H_{ik} = \frac{1}{\sqrt{J_i J_k}} \left[1 + \alpha \tanh\alpha - \alpha \coth\alpha - \frac{2\lambda_k^{(0)}}{\lambda_i^{(0)} - \lambda_k^{(0)}} \right] \frac{\lambda_i^{(0)}}{\lambda_i^{(0)} - \lambda_k^{(0)}} \quad (6.1)$$

$$H_{ki} = \frac{1}{\sqrt{J_i J_k}} \left[1 - \alpha \tanh\alpha + \alpha \coth\alpha - \frac{2\lambda_i^{(0)}}{\lambda_k^{(0)} - \lambda_i^{(0)}} \right] \frac{\lambda_k^{(0)}}{\lambda_k^{(0)} - \lambda_i^{(0)}} \quad (6.2)$$

$$(i = 0, 2, 4, \ldots; k = 1, 3, 5, \ldots)$$

With the use of these matrix elements we can calculate the quadratic corrections to the decrement, summing the series (3.10) of perturbation theory. Fig. 3 shows $\lambda_0^{(2)}$, $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ as functions of the wave number. The values of $\lambda_0^{(2)}$ agree well with the results of numerical calculation [7].

In the case of flow with a cubic profile the matix elements are expressed by very unwieldy equations, which are not given here. Figs. 4 and 5 show the quadratic corrections to the first five decrements as functions of the wave number, and Fig.6 shows as an example

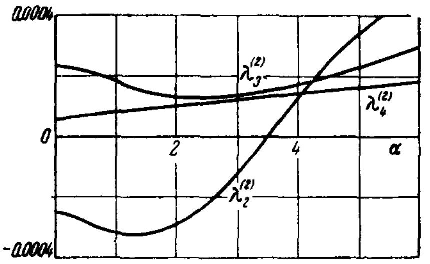

Fig. 5

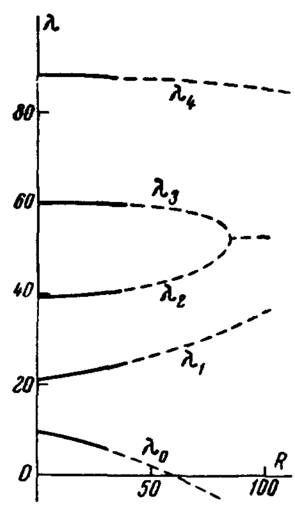

Fig. 6

the spectrum of decrements for $\alpha = 1$. The dashes correspond to extrapolation according to the quadratic correction. The value of the Reynolds number at which confluence of λ_2 and λ_3 occurs with the generation of complex conjugate decrements was found approximately by use of the equation of the theory of intersection (5.8), where $R_0 = 0$ was used. The lowest level λ_0 can be extended to intersect the axis, and thus we can estimate the critical Reynolds number, determining the boundary of stability with respect to monotonous perturbations. Such extrapolation is clearly associated with a known risk. However the conclusion of the existence of monotonous instability of flows with a cubic profile does not involve any doubt. This conclusion was also obtained earlier (by other methods) in the investigation of the stability of steady convective motion [12]. Besides, the existence of monotonous instability is confirmed also by the results of numerical calculations on an electronic computer; they will be published later. This instability distinguishes flow with a cubic profile from Couette flow where, as was shown in a general form in [13], monotonous perturbations always decay.

BIBLIOGRAPHY

1. Lin, C.C., Teoriia gidrodinamicheskoi ustoichivosti (Theory of Hydrodynamic Stability). Foreign Literature Publishing House, Moscow, 1958.
2. Schlichting, H., Vozniknovenie turbulentnosti (The Origin of Turbulence). Foreign Literature Publishing House, Moscow, 1962.
3. Grohne, D., Über das Spectrum bei Eigenschwingungen ebener Laminarströmungen. ZAMM, Bd.34, № 8/9, 1954.
4. Jahnke, E. and Emde, F., Tablitsy funktsii (Tables of Functions). Fizmatgiz, Moscow, 1959.
5. Dolph, C.L. and Lewis, D.C., On the application of infinite systems of ordinary differential equations to perturbations of plane Poiseuille flow. Quart.Appl.Math., Vol.16, № 2, 1958.
6. Szablewski, W., Das Verhalten wellenförmiger Störungen einer ebenen Poiseuilleschen Strömung bei kleinem Parameter $\alpha \cdot Re$. ZAMM, Bd.42, № 4/5, 1962.
7. Gallagher, A.P. and Mercer, A., McD., On the behavior of small disturbances in plane Couette flow. J.Fluid Mech., Vol.13, № 1, 1962.
8. Landau, L.D. and Lifshits, E.M., Kvantovaia mekhanika (Quantum Mechanics) Fizmatgiz, Moscow, 1963.
9. Shliomis, M.I., O kolebatel'noi konvektivnoi neustoichivosti provodiashchei zhidkosti v magnitnom pole (On the oscillatory convective instability of a conducting fluid in a magnetic field). PMM Vol.28, № 4, 1964.
10. Keldysh, M.V., O sobstvennykh znacheniiakh i sobstvennykh funktsiiakh nekotorykh klassov nesamosopriazhennykh uravnenii (On the eigenvalues and eigenfunctions of some classes of non-self-adjoint equations). Dokl.Akad.Nauk SSSR, Vol.77, № 1, 1951.
11. Gershuni, G.Z., Ob ustoichivosti ploskogo konvektivnogo dvizheniia zhidkosti (On the stability of plane convective motion of a fluid). Zh.tekh,Fiz., Vol.23, № 10, 1953.
12. Gershuni, G.Z.,and Zhukhovitskii, E.M., O dvukh tipakh neustoichivosti konvektivnogo dvizheniia mezhdu parallel'nymi vertikal'nymi ploskostiami (On two types of instability of convective motion between parallel vertical plates). Izv.vyssh.ucheb.Zaved.(Fizika), № 4,1958.
13. Dikii, L.A., ob ustoichivosti ploskoparallel'nogo techeniia Kuetta (On the stability of plane-parallel Couette flow). PMM Vol.28, №2, 1964.

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